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A method for obtaining symmetrized representations of SU(2) × SU(2) and the rotation group SO(4)

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Abstract. A formula is obtained by which any symmetrized power of $[j_1, j_2]$ may be expressed in terms of symmetrized powers of representations belonging to lower j values. In particular formulae are obtained for $[\frac{1}{2}, J]\otimes(2)$ and $[1, J]\otimes(2)$. The method is generalized to cover groups of the form $\mathrm{SU(2)} \times \mathrm{SU(2)} \times \ldots$ (r times). This problem has application to the L–S coupling shell model of nuclear physics.

1. Introduction

In a previous paper (Gard and Backhouse 1974, to be referred to as I) methods were obtained for symmetrizing the UIR's (unitary irreducible representations) of the group SU(2) or, equivalently, the three-dimensional rotation group SO(3). The method has now been extended to solve the problem of symmetrizing the UIR's of SU(2) × SU(2) or, equivalently, the four-dimensional rotation group SO(4). The main application of this work is to the L–S coupling shell model in nuclear physics. If the nuclear forces do not depend strongly on the spins, the total wavefunction may be written as the product of an orbital wavefunction and a function of the spin and charge variables. Due to the symmetry of the Hamiltonian in the real space variables, the orbital wavefunction will have a definite symmetry with respect to the interchange of nucleons. Hence, in order that the total wavefunction should be antisymmetric, the charge–spin function must have the conjugate symmetry. More details of this problem may be found in Hamermesh (1964) and Lomont (1957).

Denote the UIR's of SU(2) by $[j]$, $j = \frac{1}{2}, 1, \ldots$. The inner Kronecker product of UIR's of SU(2) has the simple reduced form

$$[j_1][j_2] \equiv \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} [j]. \tag{1.1}$$

This notation may be extended to write the UIR's of SU(2) × SU(2) as

$$[j_1, j_2] \equiv [j_1] \times [j_2]. \tag{1.2}$$

Since the covering group of SO(4) is SU(2) × SU(2), the results found in this paper also apply to the UIR's of SO(4). The relationship between the parameters of SO(4) and SU(2) × SU(2) is to be found in Talman (1960).

Now consider the nth inner Kronecker power of $[j_1, j_2]$ which is denoted by $[j_1, j_2]^n$. The symmetric group, S_n, acts on the carrier space of this UR (unitary representation) by permuting the basis elements of the $[j_1, j_2]$'s. Under this action the space decomposes into $(\mathrm{SU}(2)\times\mathrm{SU}(2))\times \mathrm{S}_n$-invariant subspaces Ω^ν, where $[\nu]$ is the UIR of S_n corresponding to the partition $(\nu)=(\nu_1,\nu_2,\ldots,\nu_d)$, $\nu_1\geqslant\nu_2\geqslant\ldots\geqslant\nu_d>0$, of the positive integer n. The space Ω^ν carries the UR $d_\nu[j_1,j_2]\otimes(\nu)$, where $d_\nu=\dim[\nu]$, and hence there is a direct sum decomposition

$$[j_1,j_2]^n\equiv\bigoplus_\nu d_\nu[j_1,j_2]\otimes(\nu). \tag{1.3}$$

The UR $[j_1,j_2]\otimes(\nu)$ is called the symmetrized power of $[j_1,j_2]$ corresponding to the partition (ν) of n. More details of this decomposition may be found in Boerner (1970).

The plan of the paper is as follows. In § 2 we derive the main result of this paper by which any symmetrized power of $[j_1,j_2]$ may be expressed in terms of symmetrized powers of representations belonging to lower j values (see theorems (2.2), (2.3)). In particular we obtain formulae for $[\frac{1}{2},J]\otimes(2)$ and $[1,J]\otimes(2)$. In § 3 we generalize the method to obtain a similar result for the UIR's of $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\ldots(r$ times) (see theorem (3.1)). We also note a generalization of the theorem relating symmetrized representations of SU(2) given in I.

2. General formula for $[j_1,j_2]\otimes(\nu)$

Let $\chi(j_1,j_2)$ denote the character of $[j_1,j_2]$, then for a rotation (θ_1,θ_2), $\chi(j_1,j_2)$ takes the value

$$(\mathrm{e}^{\mathrm{i}j_1\theta_1}+\mathrm{e}^{\mathrm{i}(j_1-1)\theta_1}+\ldots+\mathrm{e}^{-\mathrm{i}j_1\theta_1})(\mathrm{e}^{\mathrm{i}j_2\theta_2}+\ldots+\mathrm{e}^{-\mathrm{i}j_2\theta_2}). \tag{2.1}$$

But symmetrizing the character $\chi(j_1,j_2)$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is equivalent to symmetrizing the representation

$$(\theta_1,\theta_2)\mapsto(\mathrm{e}^{\mathrm{i}j_1\theta_1}\oplus\mathrm{e}^{\mathrm{i}(j_1-1)\theta_1}\oplus\ldots\oplus\mathrm{e}^{-\mathrm{i}j_1\theta_1})(\mathrm{e}^{\mathrm{i}j_2\theta_2}\oplus\ldots\oplus\mathrm{e}^{-\mathrm{i}j_2\theta_2}) \tag{2.2}$$

of $\overline{\mathrm{SO}(2)}\times\overline{\mathrm{SO}(2)}$ (the double groups of SO(2)). Having carried through the symmetrization, we revert to the idea of characters to obtain a reduction of the representation. As in I we need the following theorem:

Theorem (2.1). Let L and M be representations of the same group G. Let (ν) be a partition of n and let $(\mu_1),(\mu_2)$ be partitions of n_1,n_2 respectively, where $n=n_1+n_2$. Then

$$(L\oplus M)\otimes(\nu)\equiv\bigoplus_{n_1,n_2,\mu_1,\mu_2}\sigma(\nu;\mu_1,\mu_2)[L\otimes(\mu_1)][M\otimes(\mu_2)]$$

where the direct sum is taken over all partitions of n as $n=n_1+n_2$ and for each such partition $\sigma(\nu;\mu_1,\mu_2)$ is the frequency of the representation $[\mu_1]\times[\mu_2]$ in $[\nu]\downarrow S_{n_1}\times S_{n_2}$.

Denote the linear characters of $\overline{\mathrm{SO}(2)}\times\overline{\mathrm{SO}(2)}$ by $\psi_p\phi_q$ so that

$$\psi_p\phi_q(\theta_1,\theta_2)=\mathrm{e}^{\mathrm{i}p\theta_1}\,\mathrm{e}^{\mathrm{i}q\theta_2}. \tag{2.3}$$

In order to find a step-up procedure for symmetrizing representations of $\mathrm{SU}(2)\times\mathrm{SU}(2)$, we apply theorem (2.1) to representations of $\overline{\mathrm{SO}(2)}\times\overline{\mathrm{SO}(2)}$, taking

$$\begin{aligned}L&=(\psi_j\oplus\psi_{j-1}\oplus\ldots\oplus\psi_{j-p})\times(\phi_J\oplus\phi_{J-1}\oplus\ldots\oplus\phi_{-J})\\&=[\psi_{j-p/2}(\psi_{p/2}\oplus\ldots\oplus\psi_{-p/2})]\times(\phi_J\oplus\ldots\oplus\phi_{-J})\end{aligned} \tag{2.4}$$

$$\begin{aligned}M&=(\psi_{j-p-1}\oplus\ldots\oplus\psi_{-j})\times(\phi_J\oplus\ldots\oplus\phi_{-J})\\&=[\psi_{-\frac{1}{2}(p+1)}(\psi_{j-\frac{1}{2}(p+1)}\oplus\ldots\oplus\psi_{-j+\frac{1}{2}(p+1)})]\times(\phi_J\oplus\ldots\oplus\phi_{-J})\end{aligned} \tag{2.5}$$

where $p < j$. It follows that

$$L\otimes(\mu_1) = (\psi_{j-p/2})^{n_1}\{[p/2, J]\otimes(\mu_1)\downarrow \overline{SO(2)}\times\overline{SO(2)}\} \tag{2.6}$$

$$M\otimes(\mu_2) = (\psi_{-\frac{1}{2}(p+1)})^{n_2}\{[j-\tfrac{1}{2}(p+1), J]\otimes(\mu_2)\downarrow \overline{SO(2)}\times\overline{SO(2)}\}. \tag{2.7}$$

Hence

$$\begin{aligned}[j, J]\otimes(\nu)\downarrow \overline{SO(2)}\times\overline{SO(2)} \\ &\equiv (L\oplus M)\otimes(\nu) \\ &\equiv \bigoplus \sigma(\nu;\mu_1,\mu_2)\psi_{jn_1-\frac{1}{2}(n_2+np)}\{[\tfrac{1}{2}p, J]\otimes(\mu_1) \\ &\quad \times[j-\tfrac{1}{2}(p+1), J]\otimes(\mu_2)\}\downarrow \overline{SO(2)}\times\overline{SO(2)}.\end{aligned} \tag{2.8}$$

As in I, the symmetry of the problem has been lost by our choice of L and M. If instead we take

$$\begin{aligned}L &= (\psi_{-j+p}\oplus\ldots\oplus\psi_{-j})\times(\phi_J\oplus\ldots\oplus\phi_{-J}) \\ &= [\psi_{-j+p/2}(\psi_{p/2}\oplus\ldots\oplus\psi_{-p/2})]\times(\phi_J\oplus\ldots\oplus\phi_{-J})\end{aligned} \tag{2.9}$$

$$\begin{aligned}M &= (\psi_j\oplus\ldots\oplus\psi_{-j+p+1})\times(\phi_J\oplus\ldots\oplus\phi_{-J}) \\ &= [\psi_{\frac{1}{2}(p+1)}(\psi_{J-\frac{1}{2}(p+1)}\oplus\ldots\oplus\psi_{-j+\frac{1}{2}(p+1)})]\times(\phi_J\oplus\ldots\oplus\phi_{-J})\end{aligned} \tag{2.10}$$

we obtain the result

$$\begin{aligned}[j, J]\otimes(\nu)\downarrow \overline{SO(2)}\times\overline{SO(2)} \\ &\equiv \bigoplus \sigma(\nu;\mu_1,\mu_2)\psi_{-jn_1+\frac{1}{2}(n_2+np)} \\ &\quad \times\{[\tfrac{1}{2}p, J]\otimes(\mu_1)[j-\tfrac{1}{2}(p+1), J]\otimes(\mu_2)\}\downarrow \overline{SO(2)}\times\overline{SO(2)}.\end{aligned} \tag{2.11}$$

Adding (2.8) and (2.11) we obtain a direct sum of terms of the form

$$[(\psi_p\oplus\psi_{-p})(\psi_{j_1}\oplus\ldots\oplus\psi_{-j_1})]\times(\phi_{J_1}\oplus\ldots\oplus\phi_{-J_1}). \tag{2.12}$$

As in I we define an operator $T(p, q)$, where

$$T(p, q) = T(p, 0)+T(0, q). \tag{2.13}$$

Each component acts just as for SU(2) so that

$$\begin{aligned}T(p, 0)[j, J] &= [j+p, J]\oplus[j-p, J] \\ T(0, q)[j, J] &= [j, J+q]\oplus[j, J-q]\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}[-j, J] &= -[j-1, J] \\ [j, -J] &= -[j, J-1].\end{aligned} \tag{2.15}$$

Hence we obtain the following result.

Theorem (2.2).

$$2[j, J]\otimes(\nu) \equiv \bigoplus \sigma(\nu;\mu_1,\mu_2)T(-jn_1+\tfrac{1}{2}(n_2+np), 0)$$

where $T(0, 0)$ is twice the identity operator.

If we had applied the same analysis to the set of elements $\{\phi_q\}$ we should have obtained the result:

Theorem (2.3).

$$2[j,J]\otimes(\nu) \equiv \bigoplus \sigma(\nu;\mu_1,\mu_2)T(0,\ -Jn_1+\tfrac{1}{2}(n_2+np))\{[j,\tfrac{1}{2}p]\otimes(\mu_1)[j,J-\tfrac{1}{2}(p+1)]\otimes(\mu_2)\}.$$

These are very useful theorems since they lead to expressions for $[j,J]\otimes(\nu)$ in terms of representations of $SU(2)\times SU(2)$ belonging to lower j,J values. Hence we can obtain any symmetrized representation by a step-up procedure. We note the following special cases of theorem (2.2).

Corollary (2.4).

(i) If $p=0$

$$2[j,J]\otimes(\nu) \equiv \bigoplus \sigma(\nu;\mu_1,\mu_2)T(jn_1-\tfrac{1}{2}n_2,0)\{[0,J]\otimes(\mu_1)[j-\tfrac{1}{2},J]\otimes(\mu_2)\}$$

(ii) If j is integral and $p=j-1$

$$2[j,J]\otimes(\nu) \equiv \bigoplus \sigma(\nu;\mu_1,\mu_2)T(\tfrac{1}{2}j(n_2-n_1)-\tfrac{1}{2}n_1,0)\{[\tfrac{1}{2}(j-1),J]\otimes(\mu_1)[j/2,J]\otimes(\mu_2)\}$$

(iii) If j is half-integral and $p=j-\tfrac{1}{2}$

$$2[j,J]\otimes(\nu) \equiv \bigoplus \sigma(\nu;\mu_1,\mu_2)T(\tfrac{1}{2}(j+\tfrac{1}{2})(n_2-n_1),0)\{[\tfrac{1}{2}(j-\tfrac{1}{2}),J]\otimes(\mu_1)[\tfrac{1}{2}(j-\tfrac{1}{2}),J]\otimes(\mu_2)\}.$$

Corollary (2.5).

$$2[j,J]\otimes(n) \equiv \bigoplus_{n_1+n_2=n} T(-jn_1+\tfrac{1}{2}(n_2+np),0)\{[\tfrac{1}{2}p,J]\otimes(n_1)[j-\tfrac{1}{2}(p+1),J]\otimes(n_2)\}.$$

As an example of theorem (2.2) we consider the case $J=j=\tfrac{1}{2}$, then

$$2[\tfrac{1}{2},\tfrac{1}{2}]\otimes(\nu) \equiv \bigoplus \sigma(\nu;\mu,\mu')T(\tfrac{1}{2}(n_2-n_1),0)\{[0,\tfrac{1}{2}]\otimes(\mu)[0,\tfrac{1}{2}]\otimes(\mu')\}.$$

Since $[0,\tfrac{1}{2}]$ is a two-dimensional UIR, terms on the right-hand side will vanish if either (μ) or (μ') is a partition of n into more than two parts. Hence

$$2[\tfrac{1}{2},\tfrac{1}{2}]\otimes(\nu) \equiv \bigoplus \sigma(\nu;\mu,\mu')T(\tfrac{1}{2}(n_1-n_2),0)\{[0,\tfrac{1}{2}(\mu_1-\mu_2)][0,\tfrac{1}{2}(\mu_1'-\mu_2')]\}. \tag{2.16}$$

An equivalent form of equation (2.16) has been obtained independently by N B Backhouse (Backhouse and Gard 1974). As an illustration we work out the known result for $[\tfrac{1}{2},\tfrac{1}{2}]\otimes(n)$.

$$\begin{aligned}2[\tfrac{1}{2},\tfrac{1}{2}]\otimes(n) &\equiv \bigoplus_{n_1+n_2=n} T(\tfrac{1}{2}(n_1-n_2),0)\{[0,\tfrac{1}{2}n_1][0,\tfrac{1}{2}n_2]\}\\ &\equiv \bigoplus [0,\tfrac{1}{2}n_1][0,\tfrac{1}{2}n_2]T(\tfrac{1}{2}(n_1-n_2),0)[0,0]\\ &\equiv \bigoplus \{[0,\tfrac{1}{2}n]\oplus\ldots\oplus[0,\tfrac{1}{2}(n_1-n_2)]\}\{[\tfrac{1}{2}(n_1-n_2),0]-[\tfrac{1}{2}(n_1-n_2)-1,0]\}\\ &\equiv \left(\bigoplus_{n_1+n_2=n}[\tfrac{1}{2}(n_1-n_2),\tfrac{1}{2}(n_1-n_2)]\right)-[-\tfrac{1}{2}n-1,\tfrac{1}{2}n].\end{aligned}$$

Hence

$$[\tfrac{1}{2},\tfrac{1}{2}]\otimes(n) \equiv [\tfrac{1}{2}n,\tfrac{1}{2}n]\oplus[\tfrac{1}{2}n-1,\tfrac{1}{2}n-1]\oplus\ldots\oplus\begin{cases}[0,0]\\ [\tfrac{1}{2},\tfrac{1}{2}]\end{cases} \tag{2.17}$$

the last term depending on whether n is even or odd. In the above analysis we used the result that

$$[j_1, J_1]\{T(p,0)[j_2, J_2]\} \equiv T(p,0)\{[j_1, J_1][j_2, J_2]\} \tag{2.18}$$

which is easily deducible.

As an example of corollary (2.5) we consider the case $n = 2$, then

$$2[j, J]\otimes(2) \equiv T(1,0)\{[j-\tfrac{1}{2}, J]\otimes(2)\}\oplus T(2j,0)\{[0,J]\otimes(2)\}\oplus T(j-\tfrac{1}{2},0)\{[j-\tfrac{1}{2}, J][0,J]\}.$$

For the special case $j = \frac{1}{2}$ we have

$$\begin{aligned}[\tfrac{1}{2}, J]\otimes(2) &\equiv [0,J][0,J]\oplus T(1,0)\{[0,J]\otimes(2)\}\\ &\equiv \bigoplus_{k=1,\frac{1}{2}}^{J} [0,2k-1]\oplus \bigoplus_{k=0,\frac{1}{2}}^{J} [1,2k]\end{aligned} \tag{2.19}$$

where the lower limit corresponds to J being integer or half-integer respectively. Also we have

$$2[1, J]\otimes(2) \equiv T(1,0)\{[\tfrac{1}{2}, J]\otimes(2)\}\oplus T(2,0)\{[0,J]\otimes(2)\}\oplus T(\tfrac{1}{2},0)\{[\tfrac{1}{2}, J][0,J]\}.$$

Hence

$$[1, J]\otimes(2) \equiv \bigoplus_{k=0,\frac{1}{2}}^{J} [2,2k]\oplus[0,2k]\oplus \bigoplus_{k=1,\frac{1}{2}}^{J} [1,2k-1]. \tag{2.20}$$

Clearly it is quite easy to step up to the value of $[j, J]\otimes(2)$. For $n = 3$ we have

$$[\tfrac{1}{2}, J]\otimes(3) \equiv T(\tfrac{3}{2},0)\{[0,J]\otimes(3)\}\oplus T(\tfrac{1}{2},0)\{[0,J]\otimes(2)[0,J]\}. \tag{2.21}$$

This can be worked out using results (4.10) and (4.11) of I. It is also possible to find $[\frac{1}{2}, 1]\otimes(n)$ by using result (5.3) of I.

3. General formula for $[j_1, j_2, \ldots, j_r] \otimes (v)$

Having solved the problem of symmetrizing UIR's of SU(2) × SU(2) it is now possible to find a method of symmetrizing UIR's of SU(2) × SU(2) × ... (r times). The representations of this group are denoted by $[j_1, j_2, \ldots, j_r]$ and are equivalent to the outer direct product of r UIR's of SU(2). Proceeding exactly as in § 2 we arrive at the following result:

Theorem (3.1).

$$\begin{aligned}&2[j_1, j_2, \ldots, j_r]\otimes(v)\\ &\quad\equiv \bigoplus \sigma(v;\mu_1,\mu_2)T(-j_1 n_1+\tfrac{1}{2}(n_2+np),0,\ldots,0)\\ &\quad\quad\times\{[\tfrac{1}{2}p, j_2, \ldots, j_r]\otimes(\mu_1)[j_1-\tfrac{1}{2}(p+1), j_2, \ldots, j_r]\otimes(\mu_2)\}\end{aligned}$$

where $T(\mathbf{0})$ is twice the identity operator.

In principle this result allows us to build up to any symmetrized power

$$[j_1, j_2, \ldots, j_r]\otimes(v).$$

We note the following result:

Corollary (3.2).

$$
\begin{aligned}
&2[j_1, j_2, \ldots, j_r]\otimes(n)\\
&\qquad\equiv \bigoplus T(-j_1 n_1+\tfrac{1}{2}(n_2+np), 0, \ldots, 0)\{[\tfrac{1}{2}p, j_2, \ldots, j_r]\otimes(n_1)\\
&\qquad\quad\times[j_1-\tfrac{1}{2}(p+1), j_2, \ldots, j_r]\otimes(n_2)\}.
\end{aligned}
$$

By analogy with theorems (2.2) and (2.3) we obtain $(r-1)$ other results by applying the transposition $(1k)$, $2 \leqslant k \leqslant r$, to all r-tuples in the statement of the above two results. Note that the equivalent results for SU(2) itself are more general than those given in I.

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References

Backhouse N B and Gard P 1974 *Proc. 3rd Int. Colloqu. on Group Theoretical Methods in Physics* (Marseilles: CNRS) to be published

Boerner H 1970 *Representation of Groups* (Amsterdam: North-Holland)

Gard P and Backhouse N B 1974 *J. Phys.* A: *Math., Nucl. Gen.* 7 1793–803

Hamermesh M 1964 *Group Theory* (Reading, Mass.: Addison-Wesley)

Lomont J S 1959 *Applications of Finite Groups* (New York: Academic Press)

Talman J 1960 *Special Functions: A Group Theoretic Approach* (New York: Benjamin)